

Quantum corrections to static solutions of phi-in-quadro and Sin-Gordon models via generalized zeta-function

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Abstract

A general algebraic method of quantum corrections evaluation is presented. Quantum corrections to a few classical solutions (kinks and periodic) of Ginzburg-Landau (phi-in-quadro) and Sin-Gordon models are calculated in arbitrary dimensions. The Green function for heat equation with a soliton potential is constructed by means of Laplace transformation theory and Hermit equation for the Green function transform. The generalized zeta-function is used to evaluate the functional integral and quantum corrections to mass in quasiclassical approximation.

1 Introduction.

1.1 General remarks.

We consider one-dimensional field theory, based on nonlinear Klein-Gordon equations, arising, for example of Sine-Gordon (SG) case, in kink models for crystal structure dislocations [1]. In [2] the authors study the diffusion of kinks. The Ginzburg-Landau (GL) model is very popular in different aspects of solid state physics, e.g. for magnetics [3].

1.2 Feynmann quantization of a classical field.

An attention to Feynmann quantization formalism of a classical field was recently attracted in connection with a link to a SUSY quasiclastic quantization condition [4] suggested in [5]; see, however, [6].

A class of nonlinear Klein-Gordon equations in the case of static one-dimensional solutions is reduced to

$$\phi'' - V'(\phi) = 0, \phi = \phi(x), x \in R. \quad (1.1)$$

Suppose the potential $V(\phi)$ is twice continuously differentiable; it guarantees existence and uniqueness of the equations correspondent to (1.1) Cauchy problem solution. The first integral of (1.1) is given by

$$W = \frac{1}{2}(\phi'^2 - V(\phi)), \quad (1.2)$$

where W is the integration constant. The equation (1.2) is ordinary first-order differential equation with separated variables. As the phase method shows, solutions of this equations belong to the following families: constant, periodic, separatrix and the so-called "passing" one [7]. In the case of phi-in-quadro (GL, non-integrable) model, the "potential" is

$$V_{gl}(\phi) = \frac{g}{4}(\phi^2 - \frac{m^2}{g})^2,$$

while in the case of Sin-Gordon (SG) model it is

$$V_{sg}(\phi) = \frac{2m^4}{3g}(1 + \cos(\frac{1}{m}\sqrt{\frac{3g}{2}}x)). \quad (1.3)$$

We modified this last model to fit it with the first one at small values of the constant g , namely

$$V_{sg} = \frac{13m^4}{12g} + V_{gl} + O(g^2).$$

In the papers of V.Konoplich [8] quantum corrections to a few classical solutions by means of Riemann zeta-function are calculated in dimensions $d > 1$. Most interesting of them are the corrections to the kink - the separatrix solution of the field ϕ^4 (GL) model. The method of [8] is rather complicated and it is desired to simplify it, that was the main target of our previous note [9]. We applied the Darboux transformations technique with some nontrivial details missed in [8]. The suggested approach open new possibilities; for example it allows to show the way to calculate the quantum corrections to matrix models of similar structure, Q-balls [10] and periodic solutions of the models. The last problem is posed in the review [7].

The approximate quantum corrections to the solutions of the equation (1.1) are obtained via the Feynmann functional integral method evaluated by the stationary phase analog [11]. It gives the following relation

$$\exp[-\frac{S_{qu}}{\hbar}] \simeq \frac{A}{\sqrt{D}}, \quad (1.4)$$

where S_{qu} denotes quantum action, corresponding the potential $V(\phi)$, A - some quantity determined by the vacuum state at $V(\phi) = 0$, and $\det D$ is the determinant of the operator

$$D = -\partial_x^2 - \Delta_y + V''(\phi(x)). \quad (1.5)$$

The variable $y \in R^{d-1}$ stands for the transverse variables on which the solution $\phi(x)$ does not depend. The operator D appears while the second variational derivative of the quantum action functional (which enter the Feynmann trajectory integral) is evaluated. For the vacuum action

S_{vac} the relation of the form (1.4) is valid if S_{qu} is changed to S_{vac} and D is replaced by the "vacuum state" operator $D_0 = -\partial_x^2 - \Delta_y$. Then, the quantum correction

$$\Delta S_{qu} = S_{qu} - S_{vac}, \quad (1.6)$$

is obtained by the mentioned twice use of the formula (1.4) as

$$\Delta S_{qu} = \frac{\hbar}{2} \ln\left(\frac{\det D}{\det D_0}\right). \quad (1.7)$$

Hence, the problem of determination of the quantum correction is reduced to one of evaluation of the determinants D and D_0 ratio. The methodic of the evaluation will be presented in the following section.

1.3 The generalized Riemann zeta-function and Green function of heat equation.

The generalized zeta-function appears in many problems of quantum mechanics and quantum field theories which use the Lagrangian $\mathcal{L} = (\partial\phi)^2/2 - V(\phi)$ and it is necessary to calculate a Feynmann functional integral in the quasiclassical approximation.

The scheme is following. Let $\{\lambda_n\} = S$ be a set of all eigenvalues of a linear operator L , then, logarithm of the operator determinant is represented by the formal sum over this set

$$\ln(\det L) = \sum_{\lambda_n \in S} \ln \lambda_n, \quad (1.8)$$

where the sum in the r.h.s. is formal one, S means the operator's spectrum. The generalized Riemann zeta-function $\zeta_L(s)$ is defined by the equality

$$\zeta_L(s) = \sum_{\lambda_n \in S} \lambda_n^{-s}. \quad (1.9)$$

This definition should be interpreted as analytic continuation to the complex plane of s from the half plane $\text{Res} > \sigma$ in which the sum in (1.9) converges. Differentiating the relation (1.9) with respect to s at the point $s = 0$ yields

$$\ln(\det L) = \zeta'_L(0). \quad (1.10)$$

The generalized zeta-function (4.71) admits the representation via the diagonal g_D of a Green function of the operator $\partial_t + L$. Such representation is obtained as follows.

Let $\mathbf{r} \in \mathbb{R}$ be the set of independent variables of the operator L ; particularly, the operator D of (1.5) depends on $\mathbf{r} = (x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{d-1}$, then, at $t > 0$,

$$(\partial_t + L)g(t, \mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0) \quad (1.11)$$

and

$$g(t, \mathbf{r}, \mathbf{r}_0) = 0, \quad t < 0.$$

There is a representation in terms of the formal sun

$$g_D(t, \mathbf{r}, \mathbf{r}_0) = \sum_{\mathbf{n}} \exp[-\lambda_{\mathbf{n}} t] \psi_{\mathbf{n}}(\mathbf{r}) \psi_{\mathbf{n}}^*(\mathbf{r}_0), \quad (1.12)$$

where the normalized eigenfunctions $\psi_n(\mathbf{r})$ correspond to eigenvalues λ_n of the operator L . Put $L \equiv D$. Let us introduce the function

$$\gamma_D(t) = \int d\mathbf{r} g_D(t, \mathbf{r}, \mathbf{r}) = \sum_{\mathbf{n}} \exp[-\lambda_{\mathbf{n}} t], \quad (1.13)$$

that follows from (1.12) and normalization. The integration in (1.13) is performed either along the axis in the kink case or over the period in a case of periodic solutions. The Mellin transformation of (1.13) yields in

$$\zeta_D(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \gamma_D(t) dt, \quad (1.14)$$

where the $\Gamma(s)$ is the Euler Gamma-function.

The generalized zeta-function, defined by the relations (1.13,1.14), will be referred as the zeta-function of the operator D .

From the relation (1.13) for the function $\gamma_D(t)$ it follows an important property of multiplicity: *if the operator D is a sum of two differential operators $D = D_1 + D_2$, which depend on different variables, the following equality holds*

$$\gamma_D(t) = \gamma_{D_1}(t) \gamma_{D_2}(t). \quad (1.15)$$

We will need the value of the function $\gamma_D(t)$ for the vacuum state, when the operator $D = D_0 = -\Delta$ is equal to the d-dimensional Laplacian. In this case the formal sum in the r.h.s of (1.13) goes to d-dimensional Poisson integral

$$\gamma_{D_0}(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}} d\mathbf{k} \exp(-|\mathbf{k}|^2 t) = (4\pi t)^{-d/2}. \quad (1.16)$$

More generally, for a constant potential ν ,

$$\gamma_{D_\nu}(t) = (4\pi)^{\frac{1-d}{2}} \frac{\Gamma(s + \frac{1-d}{2})}{\Gamma(s)} \nu^{\frac{d-1-2s}{2}}. \quad (1.17)$$

The basic relation (1.7) points to a necessity of evaluation of the determinants of the operators

$$D = D_0 + u(x), \quad D_0 = -\partial_x^2 - \Delta_y + \lambda \quad (1.18)$$

where λ is a positive number and $x \in R$ is one of variables, while $\mathbf{y} \in R_{d-1}$ is a set of other variables. The operator Δ_y is the Laplace operator in d-1 dimensions, $u(x)$ is one-dimensional potential that is defined by the condition

$$V''(\phi_0(x)) = \lambda + u(x), \quad (1.19)$$

where $\phi_0(x)$ is the classical static solution of the equation of motion.

A quantum correction to the action in one-loop approximation for the classical solution $\phi(x)$ is calculated via zeta-function by the formula

$$\Delta\epsilon = -\zeta'_D(0)/2, \quad (1.20)$$

where

$$\zeta_D(s) = M^{2s} \int_0^\infty \gamma(t) t^{s-1} dt / \Gamma(s); \quad (1.21)$$

here $\Gamma(s)$ is the Euler gamma function and M is a mass scale.

The function $\gamma(t)$ in the Mellin integral (1.21) is expressed via the Green functions difference $G(x, y, t; x_0, y_0, t_0)$ and $G_0(x, y, t; x_0, y_0, t_0)$ of $\partial_t + D$ and $\partial_t + D_0$ in the following way; let

$$g(x, t) = G(x, y, t; x_0, y_0, 0) - G_0(x, y, t; x_0, y_0, 0),$$

due to the translational invariance along y of operators D and D_0 the function g does not depend on y ; the contraction of G_0 is necessary for deleting of ultraviolet divergence.

2 Static solutions of the SG and φ^4 model

Starting with SG model we integrate the equation (1.1) with the potential (1.3) arriving at the first-order differential equation with the parameter W (1.2).

$$(\varphi')^2 = \frac{4m^4}{3g} (1 + \cos(\frac{1}{m} \sqrt{\frac{3g}{2}} \varphi)) + 2W. \quad (2.22)$$

The solutions are restricted and hence have the direct physical relevance, if

$$-\frac{4m^4}{3g} \leq W \leq 0, \quad (2.23)$$

that follows from phase plane analysis. If $W = 0$ the nontrivial solutions are interpreted as kink and antikink

$$\varphi(x) = \pm \sqrt{\frac{2}{3g}} \arcsin \tanh(mx) (\text{mod } \Phi), \quad \Phi = 2m\pi \sqrt{\frac{2}{3g}}, \quad (2.24)$$

while at the interval

$$-\frac{4m^4}{3g} < W < 0,$$

the solution of (2.22) yields a periodic function expressed via elliptic Jacobi function. To find it one plug $\varphi = \pm 2m \frac{2}{3g} \arcsin z$, then the equation (2.22) goes to

$$(z')^2 = m^2 (1 - z^2) (1 + \frac{3gW}{4m^4}). \quad (2.25)$$

The solution of (2.25) at the interval (2.23) is given by

$$z = k \operatorname{sn}(mx; k), \quad (2.26)$$

where

$$k = \sqrt{1 + \frac{3gW}{4m^4}} \quad (2.27)$$

is the module of the elliptic function. Hence

$$W = \frac{4(k^2 - 1)m^4}{3g}. \quad (2.28)$$

Finally

$$\varphi = \pm 2m \frac{2}{3g} \arcsin ksn(mx; k) \pmod{\Phi}. \quad (2.29)$$

The class of restricted solutions contains also

$$\varphi = 0 \pmod{\Phi}; W = -\frac{3m^4}{3g} \quad (2.30)$$

and

$$\varphi = \pm \pi m \sqrt{\frac{2}{3g}} \pmod{\Phi}; W = 0. \quad (2.31)$$

Other static solutions are obtained by shifts

$$x \rightarrow x + x_0, \quad \varphi \rightarrow \varphi + \Phi,$$

that follows from Klein-Gordon equation invariance and SG equation potential periodicity.

In the case of static solutions of the φ^4 model the potential and the Lagrangian are determined by the formulas

$$\begin{aligned} V(\varphi) &= g\varphi^4/4 - m^2\varphi^2/2, \\ L &= -(\varphi')^2 - g\varphi^4/4 + m^2\varphi^2/2, \end{aligned}$$

therefore the equation of motion has the form

$$\varphi''(x) + m^2\varphi - g\varphi^3 = 0, \quad (2.32)$$

that yields (1.2) in the form

$$(\varphi')^2 = \frac{g}{2}(\varphi^2 - \frac{m^2}{g})^2 + 2W. \quad (2.33)$$

Its restricted solutions, as it follows from phase plane analysis, exist if

$$-\frac{m^2}{4g} \leq W \leq 0. \quad (2.34)$$

The separatrix ($W=0$) solution of (2.33) is the kink/antikink

$$\varphi_0 = \pm \sqrt{\frac{2}{g}} b \tanh(bx), \quad b = \frac{m}{\sqrt{2}}. \quad (2.35)$$

While inside the interval (2.34) the equation (2.33) is expressed in terms of the elliptic Jacobi sinus

$$\varphi = \pm \sqrt{\frac{2}{g}} kbsn(bx; k), \quad b = \frac{m}{\sqrt{1+k^2}}, \quad 0 < k < 1. \quad (2.36)$$

The constant W is given by

$$W = -\left(\frac{1-k^2}{1+k^2}\right)^2 \frac{m^4}{4g}. \quad (2.37)$$

The family of the restricted solutions contains also the constant vacuum ones ($W=0$)

$$\varphi = \pm \frac{m}{\sqrt{g}}. \quad (2.38)$$

After the substitution of (2.35) into (1.19) we obtain the following potential $u(x)$:

$$u(x) = -6b^2/ch^2(bx), \quad (2.39)$$

with the meaning of the constant $b = m/\sqrt{2}$. As a result the two-level reflectionless potential of one-dimensional Schrödinger equation $-\partial_x^2 + u(x)$ appears. Eigenvalues and the normalized eigenfunctions of which are correspondingly (its numeration is chosen from above to lowercase).

$$\begin{aligned} \lambda_1 &= -b^2, & \psi_1(x) &= \sqrt{3b/2} \sinh(bx)/\cosh^2(bx); \\ \lambda_2 &= -4b^2, & \psi_2(x) &= \sqrt{3b/2} \cosh(bx). \end{aligned}$$

3 The energy of classic static solutions of SG and ϕ^4 models

Let us evaluate the energy of the nontrivial static solutions of both models via the energy density definitions

$$e(x) = \int_{-\infty}^{\infty} ((\phi')^2/2 + V(\phi)) dx, \quad (3.40)$$

for kinks and, in a case of periodic solutions,

$$E = 2 \int_0^l e(x) dx, \quad (3.41)$$

the constant "l" is the period of the solution.

For SG kink:

$$E_k = \frac{16m^2}{g}, \quad (3.42)$$

for a periodic soliton

$$E_p = \frac{8m^2}{g} [(1 - k^2)K + 2E], \quad (3.43)$$

where $K(k), E(k)$ - complete elliptic Legendre integrals.

4 The generalized zeta-function as analytic function.

Let us consider the Laplace transform of the one-dimensional Green function defined in the Sec.1 by the relations (1.11,1.12)

$$g_L(t, x, x_0) = \frac{1}{2\pi i} \int_l \hat{g}_L(p, x, x_0) e^{pt}. \quad (4.44)$$

where the integration is performed along the contour in complex p -plane that contain all the singularities of the integrand . The integrand function $\hat{g}_L(p, x, x_0)$ may be considered as the Green function of the spectral equation

$$(L + p)\hat{g}_L(p, x, x_0) = \delta(x - x_0). \quad (4.45)$$

The corresponding transform of $\gamma(t)$ is denoted as

$$\hat{\gamma}(p) = \int \hat{g}_L(p, x, x) dx, \quad (4.46)$$

the inverse transform of (4.46) looks as (4.44) with the same contour of integration

$$\gamma_L(t) = \int_l \hat{\gamma}(p) e^{pt} dp. \quad (4.47)$$

The inverse of the formula (4.46) may be considered as a base for analytic continuations. It is known from the general properties Laplace transform of a distribution that are zero at $t < 0$ [14], so that the functions (4.46) and $\hat{g}_L(p, x, x_0)$ are analytic function at $\text{Re } p > \sigma_0$.

It also shows that no necessity to solve the equation for the Green function (1.11), it is enough to solve the spectral problem (4.45)

In a spirit of Hermit approach, see, e.g. [15], the function $\hat{g}_L(p, x, x)$ is a solution of bilinear equation

$$2GG'' - (G')^2 - 4(u(x) - p)G^2 + 1 = 0, \quad G(p, x) = \hat{g}_L(p, x, x) \quad (4.48)$$

which in a case of reflectionless and finite-gap solutions is solved more effectively than (4.45).

In the case of the periodic solution of the phi-in-quadro model the potential has a "cnoidal" form

$$u(x) = -6k^2 b^2 cn^2(bx; k) + (5k^2 - 1)b^2. \quad (4.49)$$

This potential differs from second Lamé's equation potential by the constant hence its spectrum is two-gap one.

It is possible to cover all necessary classes of solutions of both models (cases A, B for ϕ^4 and C, D for SG) via the universal representation by means of polynomials (in p) P, Q

$$G(p, x) = P(p, z)/2\sqrt{Q(p, z)}, \quad (4.50)$$

where

$$z = \text{sech}^2(bx)$$

for kinks A, C, and

$$z = cn^2(bx; k)$$

for the periodic B, D.

In the ϕ^4 (A, B case) the function $P(p)$ is linear in p polynomial while $Q(p)$ is one of the third order. Plugging (4.50) yields

$$b^2(\rho(z)(2PP'' - (P')^2) + \rho'(z)PP') - (p + u(z))P^2 + Q = 0, \quad (4.51)$$

the primes denote derivatives with respect to z , while

$$\rho = \begin{cases} z^2(1-z), & \text{cases } A, B; \\ z(1-z)(1-k^2+k^2z), & \text{cases } C, D. \end{cases} \quad (4.52)$$

$$u(z) = \begin{cases} b^2(1-2z), & \text{case } (A) \\ b^2(2k^2-1-2k^2z), & \text{case } (B) \\ 2b^2(2-3z), & \text{case } (C) \\ b^2(5k^2-6k^2z), & \text{case } (D) \end{cases} \quad (4.53)$$

Let us start with the case (A). The form of the polynomial is determined from well-known facts of the reflectionless potentials theory.

$$Q = p^2(p + b^2). \quad (4.54)$$

The case (B)

$$P = p + P_1(z), \quad Q = p^3 + q_2 p^2 + q_1 p + q_0 m \quad (4.55)$$

includes (A) in a sense that for the case $q_2 = b^2$, $q_1 = 0$, $q_0 = 0$.

Substituting (4.55) into (4.51) gives for each power of $p = 0, 1, 2$

$$\begin{aligned} -2P_1(z) - u(z) + q_2 &= 0, \\ b^2(2\rho(z)P_1'' + \rho'(z)P_1') - P_1^2 - 2u(z)P_1 + q_1 &= 0, \\ b^2(\rho(z)(2P_1P_1'' - P_1^2) + \rho'(z)P_1P_1' - u(z)P_1^2 + q_0) &= 0. \end{aligned} \quad (4.56)$$

respectively. In the case (A) it gives

$$P_1 = b^2 z. \quad (4.57)$$

For the case (B) we begin from

$$P_1 = k^2 b^2 z, \quad (4.58)$$

that is the result of substitution of P_1 from the first equation of (4.56) into the second one. Next, from the third equation yields

$$Q = p(p^2 + (1 - k^2)b^2)(p - k^2 b^2), \quad (4.59)$$

with the simple roots p_i .

Going to the cases (C,D), generally

$$P = p^2 + P_1(z)p + P_2(z), \quad Q = p^5 + q_4 p^4 + q_3 p^3 + q_2 p^2 + q_1 p + q_0, \quad (4.60)$$

while in the particular case of (C)

$$q_0 = 0, \quad q_1 = 0, q_2 = 36b^6, \quad q_3 = 33b^4, q_5 = 10b^2.$$

A substitution of (4.60) into the Hermit equation splits in the system

$$\begin{aligned} -2P_1 - u + q_2 &= 0, \\ -2P_2 - P_1^2 - 2uP_1 + b^2(2\rho P_1'' + \rho' P_1') + q_3 &= 0, \\ b^2(\rho(2P_2'' + 2P_1P_1'' - (P_1')^2) + \rho'(P_2' + P_1P_1')) - 2P_1P_2 - u(2P_2 + P_1^2) + q_2 &= 0, \\ b^2(2\rho(2P_1'P_2 + P_1P_2' + P_1P_2'') + \rho'(P_1P_2') + P_1'P_2)) - P_2^2 - 2uP_1P_2 + q_1 &= 0, \\ b^2(\rho(2P_2P_2'' - P_2'^2)) + \rho'P_2^2 + q_0 &= 0. \end{aligned} \quad (4.61)$$

The arguments in (4.61) are omitted. Solving the system yields for the case D

$$P_1(z) = \alpha z + \beta, \quad P_2(z) = az^2 + bz + c, \quad (4.62)$$

where

$$\begin{aligned} \alpha &= 3k^2 b^2, \\ \beta &= \frac{(1-5k^2)b^2 - q_4}{2}, \\ a &= 9k^4 b^4, \\ b &= \frac{3k^2 b^2(q_4 + (1-11k^2)b^2)}{2}, \\ c &= \frac{4q_3 - q_4^2 + 2(1-5k^2)b^2 q_4 + 3(1-6k^2 + 21k^4)b^4}{8}, \end{aligned} \quad (4.63)$$

are functions only of k, b , where

$$\begin{aligned} q_0 &= 0, q_1 = -27k^2(1 - k^2)b^8, \\ q_2 &= -9(1 - 3k^2 - 3k^4 - k^6)b^6, q_3 = 3(1 + k^2 + k^4)b^4, q_4 = 5(1 + k^2)b^2. \end{aligned} \quad (4.64)$$

Finally,

$$Q = \prod_{i=1}^{i=5} (p - p_i), \quad (4.65)$$

where the polynomial Q the simple roots p_i are ordered so that $(0 \leq p_1 < p_2 < \dots < p_5)$

$$-(2\sqrt{1 - k^2 + k^4} + 1 + k^2)b^2 < -3b^2 < -3k^2b^2 < 0 < -(2\sqrt{1 - k^2 + k^4} - 1 - k^2)b^2. \quad (4.66)$$

Let us pick up the expressions determining $\hat{\gamma}(p)$:

$$\hat{\gamma}(p) = \int \frac{P(z)}{3\sqrt{Q}} dx = \frac{p^2}{2\sqrt{Q}} \int dx + \frac{p}{2\sqrt{Q}} \int (\alpha z + \beta) dx + \frac{1}{2\sqrt{Q}} \int (az^2 + bz + c) dx. \quad (4.67)$$

So we need three integrals over the period K .

$$\int_0^K dx, \int_0^K z dx, \int_0^K z^2 dx. \quad (4.68)$$

Let us go to the variable z , $dz = d(cn^2(x)) = -2cn(x)cn(x)dn(x)dx = 2sn(x)dn(x)dx$, a bit more convenient to put $y = 1 - z$, $dy = -dz$

$$\begin{aligned} \int_0^K dx &= \int_0^1 \frac{dy}{y(1-y)(1-k^2y)} = 2K(k), \\ \int_0^K sn^2(x; k) dx &= \frac{1}{2} \int_0^1 \frac{y dy}{y(1-y)(1-k^2y)} = \frac{K(k) - E(k)}{k^2}, \\ \int_0^K sn^4(x; k) dy &= \int_0^1 \frac{z^2 dz}{z(1-z)(1-k^2z)} = \frac{1}{3k^4} ((2 + k^2)K(k) - 2(1 + k^2)E(k)). \end{aligned} \quad (4.69)$$

hence

$$\begin{aligned} \hat{\gamma}(p) &= 2K(k) \frac{p^2}{2\sqrt{Q}} + \alpha \left(-\frac{K(k) - E(k)}{k^2} \right) + (\beta - \alpha) 2K(k) \frac{p}{2\sqrt{Q}} + \\ &+ \left(a \frac{(2+k^2)K(k) - 2(1+k^2)E(k)}{3k^4} + (b - 2a) \frac{K(k) - E(k)}{k^2} + (a - b + c) 2K(k) \right) \frac{1}{2\sqrt{Q}}. \end{aligned} \quad (4.70)$$

or, finally

$$\zeta(s) = M^{2s} \int_0^\infty \left(\int_l^\infty \hat{\gamma}(p) e^{pt} dp \right) t^{s-1} dt / \Gamma(s); \quad (4.71)$$

which is expressed via integrals

$$\begin{aligned} &\int_l^\infty \frac{e^{pt}}{\sqrt{Q}} dp \\ &\int_l^\infty \frac{pe^{pt}}{\sqrt{Q}} dp \\ &\int_l^\infty \frac{p^2 e^{pt}}{\sqrt{Q}} dp \end{aligned} \quad (4.72)$$

by a contour that contains all branch points of the integrands.

5 SG kinks

The results of the previous sections allow to evaluate corrections to actions for all four (A,B,C,D) cases. The results for ϕ^2 model kinks are well-known, see, e.g. [9], where a table for the dimensions $d=1,2,3,4$ is listed. These results fit the case B, which hence was strictly verified.

Let us present the formulas for the kinks of the SG model. The substitution of the expressions from (4.50,4.65,4.55,4.57) yields a divergence of the integral for (4.67) $\hat{\gamma}(p)$. To regularize the Green function let us divide its Laplace transform into two parts as

$$G(p, x) = G_c(p) + G_k(p, x), \quad (5.73)$$

where the part $G_c(p)$ is the Green function diagonal (the solution of (4.48) for a constant potential:

$$G_c = \frac{1}{2\sqrt{p+b^2}}. \quad (5.74)$$

The kink part is easily constructed via (4.50,4.65,4.55,4.57):

$$G_k = \frac{b^2 \operatorname{sech}^2(bx)}{2p\sqrt{p+b^2}}. \quad (5.75)$$

The representation (5.73) results in two contributions of the quantum corrections to the kink (antikink) mass.

The first one coincides with (5.76) for $\nu = m^2$.

$$\gamma_{D_0}(t) = (4\pi)^{\frac{1-d}{2}} \frac{\Gamma(s + \frac{1-d}{2})}{\Gamma(s)} m^{d-1-2s}, \quad (5.76)$$

and the second one is obtained by the general scheme, namely

$$\gamma_k(p) = \int_{-\infty}^{\infty} G_k(p, x) dx. \quad (5.77)$$

Plugging (5.75) into (5.77) yields

$$\gamma_k(p) = \frac{b}{p\sqrt{p+k^2}}. \quad (5.78)$$

The value of the integral $\int_{-\infty}^{\infty} \operatorname{sech}^2(bx) dx = 2/b$ is taken into account. the corresponding function (1.13) or, more exactly (4.47) will be denoted γ_L^R , the index R label the result of a renormalization provided by the division (5.73). The function may be found directly from a table ([17]) but we would explain the result as an example for further development of the renormalization procedure. Note that the cut for the radical $\sqrt{p+b^2}$ is made along the Re p -axis from $-\infty$ to $-b^2$, the branch with $\sqrt{p+b^2} = i\sqrt{|p+b^2|}$ on the upper and $\sqrt{p+b^2} = -i\sqrt{|p+b^2|}$ on the lower bounds of the cut is chosen. After the deformation of the integral contour one has for $t > 0$

$$\gamma_L^R = 1 - \frac{b}{\pi} \int_0^{\infty} \frac{\exp(\xi + b^2)t}{(\xi + b^2)\sqrt{\xi}} d\xi. \quad (5.79)$$

The formula (5.79) gives an expansion of the resolvent of the operator $\partial_t + L$ by the operator L spectrum. The direct substitution of (5.79) into (1.14) leads to a divergent integrals. However in this case a renormalization is not necessary, the integral in the r.h.s. of (5.79) is expressed in terms of the error function

$$\operatorname{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2} d\tau.$$

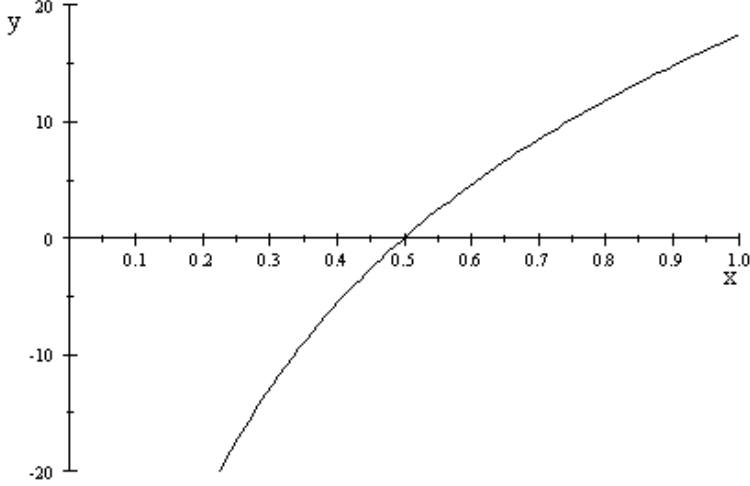


Figure 1: $d=1$, dependence of the correction $\Delta\epsilon$ to mass on the parameter m

namely, after some change of variables

$$\gamma_k(t) = \text{Erf}(b\sqrt{t}) = \frac{2b\sqrt{t}}{\sqrt{\pi}} \int_0^1 \exp[-b^2 t \tau^2] d\tau. \quad (5.80)$$

The same result gives [17]. The Mellin transform of this representation gives the following expression of zeta function of the operator L .

$$\zeta_L^R(s) = \frac{2b^{-2s}}{\sqrt{\pi}} \frac{\Gamma(s+1/2)}{\Gamma(s)} \int_0^1 \tau^{-2s-1} d\tau = -\frac{b^{-2s}}{\sqrt{\pi}} \frac{\Gamma(s+1/2)}{\Gamma(s+1)}. \quad (5.81)$$

The integral in (5.81) converged only at $\text{Res} < 0$ but gives the analytical continuation for all $s \in \mathbb{C}$ excluding the poles of $\Gamma(s+1)$.

Substituting the result (5.81) into one arising from (1.15) with account of (5.76) yields

$$\zeta_L^D(s) = -4(4\pi)^{\frac{d}{2}} m^{d-1-2s} \frac{\Gamma(s+1-d/2)}{(2s+1-d)\Gamma(s)}. \quad (5.82)$$

The condition $b = m$ is taken into account. Differentiation of (5.82) by s at the point $s = 0$ give the desired correction (1.20)

$$\begin{aligned} & -\frac{1}{2} \frac{d(\zeta_L^D(s))}{ds} = \\ & -2(4\pi)^{\frac{d}{2}} m^{d-2s-1} \frac{\Gamma(s-\frac{1}{2}d+1)}{\Gamma(s)(2s-d+1)^2} \left((d-2s-1) \left(\text{Psi}\left(-\frac{1}{2}d+s+1\right) - 2\ln m - \text{Psi}(s) \right) + 2 \right) \end{aligned} \quad (5.83)$$

next we plot the dependence of the correction $\frac{d(\zeta_L^D(0))}{ds}$ on m (Fig 3).

We would remind about the choice of the constant $g=2$ in the last sections.

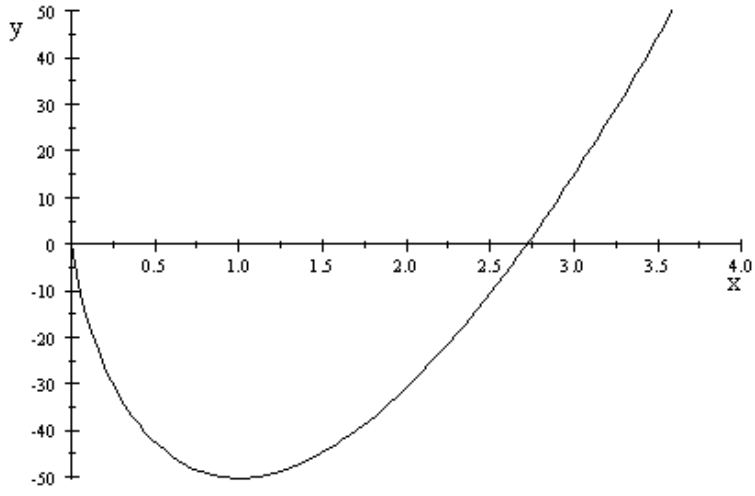


Figure 2: $d=2$, dependence of the correction $\Delta\epsilon$ to mass on the parameter m

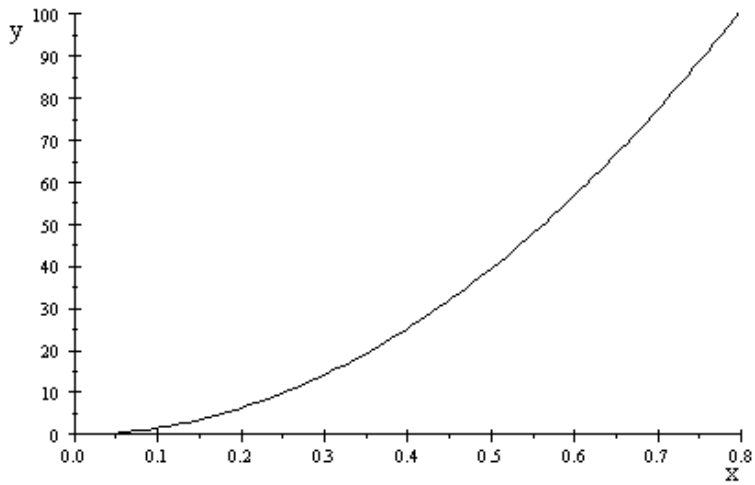


Figure 3: $d=3$, dependence of the correction $\Delta\epsilon$ to mass on the parameter m

6 Conclusion

In the next paper we admit that the potential $u(x)$ from (1.19) has the form of n -level reflectionless potential

$$u(x) = -n(n+1)b^2/\cosh^2(bx). \quad (6.84)$$

with eigenvalues $\lambda_m = -m^2b^2$, $m = 1, \dots, n$. We would note for further generalization means that this function may be considered as degenerate limit of n -gap Lamé potential of Hill equation. Such potentials correspond to higher solitonic models.

The kink case corresponds $n = 2, \lambda = n^2b^2$. The quantum correction to its action will be calculated in sec.4 from general formulas of sec.3.

Let us note that all results related to the scalar nonlinear Klein-Gordon equation may be applied directly to many-component model $\phi(x) \in \mathbb{S}^>$ with account of $\mathbb{SO}(m)$ symmetry. The equation (1.1) takes the form

$$-\phi'' + V'(|\phi|)\frac{\phi}{|\phi|} = 0. \quad (6.85)$$

The scalar operator D is defined by (1.5) goes to the matrix one

$$D = [-\partial_x^2 - \Delta_y + V'(|\phi|)\frac{\phi}{|\phi|}]I_m + [(V''(|\phi|)) - V'(|\phi|)\frac{\phi \otimes \phi}{|\phi|^3}]. \quad (6.86)$$

The technique developed in this paper is transported to quantum corrections for periodic static solutions of ϕ^4 model:

$$\phi_0(x) = \frac{km}{1+k^2}\sqrt{\frac{2}{g}}sn\left(\frac{mx}{\sqrt{1+k^2}}; k\right), \quad 0 < k \leq 1; \quad (6.87)$$

where k is a module of the elliptic function. When $k = 1$ the formula (6.87) goes to one for kink (2.35). The substitution of (6.87) into (1.19) yields in two-gap Lamé potential that is embedded in Darboux Transformations theory by chain representation [19] that give a possibility to derive the Green function analogue for this case. The results will be published elsewhere. Some recent papers open new field for applications [20, 21].

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